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UNIVERSALITY PROPERTIES OF $N = 2$ AND $N = 1$ HETEROTIC THRESHOLD CORRECTIONS

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Abstract

In the framework of heterotic compactifications, we consider the one-loop corrections to the gauge couplings, which were shown to be free of any infra-red ambiguity. For a class of $N = 2$ models, namely those that are obtained by toroidal compactification to four dimensions of generic six-dimensional $N = 1$ ground states, we give an explicit formula for the gauge-group independent thresholds, and show that these are equal within this class, as a consequence of an anomaly-cancellation constraint in six dimensions. We further use these results to compute the $(N = 2)$-sector contributions to the thresholds of $N = 1$ orbifolds. We then consider the full contribution of $N = 1$ sectors to the gauge couplings, which generically are expected to modify the unification picture. We compute such corrections in several models. We finally comment on the effect of such contributions to the issue of string unification.

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1. Introduction

In the past several years, there has been significant progress in trying to compare low-energy predictions of string theory with data [1]–[22]. String theory gives us the possibility of unifying gauge, Yukawa and gravitational interactions. The presence of supersymmetry is usually required in order to deal with hierarchy problems, although in the context of supergravity this is not automatic due to the presence of gravity (this can be verified in the framework of strings, where gravitational interactions are properly taken into account [23]). The standard folklore demands $N = 1$ supersymmetry in order for the theory to possess chiral fermions. Extended-supersymmetry ground states could also be considered provided the supersymmetry is spontaneously broken to $N = 1$ [24, 25]. Indeed, there are indications that such theories become chiral at some special points of the string moduli space.

The quantities that are most easily comparable to experimental data are effective gauge couplings of the observable sector, as well as Yukawa couplings. It is well known that the low-energy world is not supersymmetric. Thus ($N = 1$) supersymmetry has to be broken spontaneously at some scale of the order of 1 TeV (for hierarchy reasons). Although there are ways to break supersymmetry in string theory [26, 27, 28], it is fair to say that none so far has yielded a phenomenologically acceptable model. The issue of supersymmetry breaking is therefore an open problem. However, if we assume that its scale is of the order of 1 TeV and the superpartner masses are around that scale, then non-supersymmetric thresholds are not very important for dimensionless couplings (which include gauge and Yukawa couplings). Thus, it makes sense to compute them and compare them with data in the context of unbroken supersymmetry.

Threshold corrections appear in the relation between the running gauge coupling $g_i(\mu)$ of the low-energy effective field theory and the string coupling g_{string} which, assuming the decoupling of massive modes, must have the following form:

$$\frac{16\,\pi^2}{g_i^2(\mu)} = k_i \frac{16\,\pi^2}{g_{\text{string}}^2} + b_i \log \frac{M_s^2}{\mu^2} + \Delta_i \, , \tag{1.1}$$

where b_i are the usual effective field theory beta-function coefficients of the group factor G_i, and k_i is the level of the associated affine Lie algebra. The thresholds Δ_i are due to the infinite tower of string modes and can be calculated at the level of string theory. On the other hand, string unification relates the fundamental string scale $M_s \equiv \frac{1}{\sqrt{\alpha'}}$ to the Planck scale $M_P = \frac{1}{\sqrt{32\,\pi G_N}}$ and to the string coupling constant g_{string} which is associated with the expectation value of the dilaton field. At the tree level this relation reads

$$M_s = g_{\text{string}}\, M_P \ . \tag{1.2}$$

Given the fact that low-energy data, assuming the minimal supersymmetric standard model as the underlying low-energy field theory, indicate gauge unification at a scale $M_X \sim 2 \times 10^{16}$ GeV [29] which is two orders of magnitude less than the Planck scale, threshold corrections play a crucial role in string unification. Their effect has been extensively studied in the literature [18, 19, 20, 21] except for the moduli-dependent universal terms $Y(T, U)$, which

appear in the generic decomposition

$$\Delta_i = \hat{\Delta}_i - k_i\, Y\,, \tag{1.3}$$

and which have received little attention because they can be formally reabsorbed into a redefinition of $g_{\rm string}$. However, such a redefinition alters eq. (1.2) in a moduli-dependent way and consequently the relation between string unification scale and Planck mass gets modified. Following this observation, universal terms were evaluated explicitly in the context of the symmetric $Z_2 \times Z_2$ orbifold model [13], and their effect on on the unification scale of gauge couplings was shown to consist of a decrease of the order of 5 to 10% with respect to the case where these corrections are not taken into account. One of the purposes of the present article is to extend these results to more general situations. This requires the computation of one-loop gauge couplings and in particular of universal threshold corrections in more general string solutions. It is remarkable that such a computation is possible and exhibits interesting universality properties as it will appear in the sequel.

There are several procedures for computing the one-loop corrections to dimensionless couplings in string theory. The most powerful and unambiguous one was described in [12, 30]. It amounts to turning on gravitational background fields that provide the ground state in question with a mass gap Δm^2, and further background fields (magnetic fields, curvature and auxiliary F fields) in order to perform a background-field calculation of the relevant one-loop corrections.

The above procedure involves the following steps:
(i) We first regulate the infra-red by introducing a mass gap in the relevant ground state. This is done by replacing the flat four-dimensional conformal field theory with the wormhole one, $\mathbb{R}_Q \times SO(3)_{\frac{k}{2}}$ [31, 32]. The mass gap is given by $\Delta m^2 = \frac{M_s^2}{2(k+2)}$, where k is a (dimensionless) non-negative even integer.
(ii) We then turn on appropriate background fields, which are exact solutions of the string equations of motion. Such backgrounds include curvature, magnetic fields and auxiliary F fields[1].
(iii) We calculate the one-loop vacuum amplitude as a function of these background fields.
(iv) We identify these background fields as solutions of the tree-level effective action. By substituting them into the one-loop effective action and comparing with the string calculation of the free energy, we can extract the renormalization constants at one loop.

In the following we will apply the aforementioned techniques to the calculation of string loop corrections for gauge couplings. We will derive the full one-loop gauge coupling for heterotic ground states with at least $N = 1$ supersymmetry. We will in particular obtain an explicit formula for the universal part of the threshold corrections, and we will apply this formula to show how, for the whole class of $N = 2$ ground states that come from two-torus compactification of six-dimensional $N = 1$ theories, these thresholds are equal and fully determined as a consequence of an anomaly-cancellation constraint in six dimensions. We will also analyse their asymptotic behaviour and singularity structure. The universal term in particular turns out to be singularity-free and continuous inside the moduli space,

[1] These are relevant for the study of the Kähler potential renormalization. For more details see [15].

whereas some second derivatives with respect to the moduli, such as $\partial_T \partial_{\overline{T}} Y$, ..., are logarithmically divergent along enhanced-symmetry planes. We will then turn to $N = 1$ orbifold constructions in four dimensions and use the above results in order to evaluate exactly the threshold corrections originated from the $N = 2$ sectors. Although we will not have much to say concerning a general analytic formula for the ($N = 1$)-sector contributions, we will present results for two cases, namely the Z_3 and Z_4 orbifolds. Here an interesting observation is the breakdown of the usual ansatz: $\Delta_i^{N=1}$ cannot in general be decomposed as $b_i^{N=1}\,\Delta^{N=1} - k_i\,Y^{N=1}$. Using the above results, we will analyse the effect of the various thresholds on the string unification scale, and show that they actually reduce that scale. Finally, we will clarify the appearance of the Green–Schwarz term, which is another universal contribution present in $N = 1$ theories. As we will see, this one-loop correction plays no role for the issue of unification, in contrast with the universal term Y appearing in the decomposition (1.3).

We would like to stress here that the study of threshold corrections is important not only for phenomenological purposes. It will eventually be useful in the context of string-string dualities where one expects a deeper understanding of non-perturbative phenomena [33].

2. One-loop gauge couplings in heterotic ground states

As mentioned previously, an infra-red regulated version of a heterotic ground state is provided by substituting four-dimensional flat space with a suitably chosen conformal field theory, namely a $(1,0)$ supersymmetric version of the $\mathbb{R}_Q \times SO(3)_{\frac{k}{2}}$ σ-model[2] [12, 30]. This substitution preserves gauge symmetries, supersymmetry and modular invariance, and introduces curvature as well as a linear dilaton in the time direction

$$\Phi = \frac{tM_s}{\sqrt{k+2}}\,, \tag{2.1}$$

necessary for making the total central charge equal to that of flat space. This amounts to a universal mass gap for all string excitations (bosonic and fermionic) that can be read off (in the Euclidean) from the left worldsheet Hamiltonian

$$L_0 = -\frac{1}{2} + \frac{1}{4(k+2)} + \frac{p_t^2}{2} + \frac{j(j+1)}{k+2} + \cdots : \tag{2.2}$$

$\Delta m^2 = \frac{\mu^2}{2}$ with $\mu = \frac{M_s}{\sqrt{k+2}}$.

In this geometry, vertices for space-time fields such as $F^{\alpha}_{\mu\nu}$ are truly marginal world-sheet operators and therefore deformations induced by the associated background fields are exactly calculable. This allows in particular for the computation of various one-loop correlators by inserting the corresponding vertex operators. For magnetic fields we use

$$V_i^{\rm magn} \propto \left(J^3 + i : \psi^1\psi^2 :\right) \overline{J}_i\,. \tag{2.3}$$

[2] The group $SO(3)$ is required instead of $SU(2)$ for spin-statistics consistency.

This turns on a magnetic field in the third space direction; $\overline{J}_i$ is a right-moving affine current in the Cartan of the ith gauge group simple factor (picking out a single Cartan direction will be enough for our purposes), and J^3 belongs to the $SO(3)_{\frac{k}{2}}$ affine Lie algebra. There is also a gravitational perturbation generated by

$$V^{\text{grav}} \propto \left(J^3 + i : \psi^1 \psi^2 :\right) \overline{J}^3 . \tag{2.4}$$

The currents J^3, $\overline{J}^3$ and $\overline{J}_i$ are normalized so that[3]

$$J^3(z)\, J^3(0) = \frac{k}{2z^2} + \cdots , \quad \overline{J}^3(\bar{z})\, \overline{J}^3(0) = \frac{k}{2\bar{z}^2} + \cdots , \quad \overline{J}_i(\bar{z})\, \overline{J}_i(0) = \frac{k_i}{\bar{z}^2} + \cdots . \tag{2.5}$$

All the above perturbations are products of left times right Abelian currents and thus preserve conformal invariance. This implies that the new backgrounds satisfy the string equations of motion at tree level to all orders in the α' expansion.

Let $\mathcal{F}$ and $\mathcal{R}$ be constant magnetic and gravitational fields. The vacuum amplitude at one loop, i.e. the free energy, in the presence of these backgrounds can be readily calculated by performing the following Lorentz boost:

$$\begin{pmatrix} \frac{Q+I^3}{\sqrt{k/2+1}} \\ \frac{\overline{I}^3}{\sqrt{k/2}} \\ \frac{\overline{P}_i}{\sqrt{k_i}} \end{pmatrix}' = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 \\ \sinh\phi & \cosh\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{Q+I^3}{\sqrt{k/2+1}} \\ \frac{\overline{I}^3}{\sqrt{k/2}} \\ \frac{\overline{P}_i}{\sqrt{k_i}} \end{pmatrix} , \tag{2.6}$$

where $\mathcal{F}$ and $\mathcal{R}$ have to be identified with $\sinh 2\phi \sin\theta$ and $\sinh 2\phi \cos\theta$, respectively. Here $I^3, \overline{I}^3$ stand for the zero modes of the respective $SO(3)_{\frac{k}{2}}$ currents, Q is the zero mode of the $i : \psi^1 \psi^2 :$ current and $\overline{P}_i$ is the zero mode of the $\overline{J}_i$ current. We assume that the gauge background does not correspond to an anomalous $U(1)$. This case can also be treated, but is more complicated[4]. The free energy then reads:

$$\alpha'^2 F^{\text{string}}_{\text{one loop}} = \frac{1}{2(2\pi)^4} \int_{\mathcal{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2}\, D^{\text{string}}_{\text{one loop}} = \frac{1}{2(2\pi)^4} \int_{\mathcal{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \left\langle e^{-2\pi \operatorname{Im}\tau\, \delta\left(L_0 + \overline{L}_0\right)} \right\rangle , \tag{2.7}$$

with

$$\begin{aligned} \delta L_0 = \delta \overline{L}_0 \quad = \quad & \frac{\sqrt{1+\mathcal{F}^2+\mathcal{R}^2}-1}{2} \left(\frac{\left(Q+I^3\right)^2}{k+2} + \frac{1}{\mathcal{R}^2+\mathcal{F}^2} \left(\mathcal{R}\frac{\overline{I}^3}{\sqrt{k}} + \mathcal{F}\frac{\overline{P}_i}{\sqrt{2k_i}} \right)^2 \right) \\ & + \frac{Q+I^3}{\sqrt{k+2}} \left(\mathcal{R}\frac{\overline{I}^3}{\sqrt{k}} + \mathcal{F}\frac{\overline{P}_i}{\sqrt{2k_i}} \right) . \end{aligned} \tag{2.8}$$

[3]Our normalization is the one widely used in the literature; it corresponds to the situation where the highest root of the algebra has length squared $\psi^2 = 2$.

[4]Since anomalous $U(1)$'s are broken at scales comparable with the string scale, their running is irrelevant for low-energy physics.

Expanding to second order in the background fields, we find:

$$\begin{aligned}
D^{\text{string}}_{\text{one loop}} &= \langle 1 \rangle \\
&+\frac{8\pi^2(\operatorname{Im}\tau)^2\mathcal{R}^2}{k(k+2)}\left\langle \left(Q+I^3\right)^2\left(\overline{I}^3\right)^2-\frac{k}{8\pi\operatorname{Im}\tau}\left(\left(Q+I^3\right)^2+\frac{k+2}{k}\left(\overline{I}^3\right)^2\right)\right\rangle \\
&+\frac{4\pi^2(\operatorname{Im}\tau)^2\mathcal{F}^2}{k_i(k+2)}\left\langle \left(Q+I^3\right)^2\overline{P}_i^2-\frac{k_i}{4\pi\operatorname{Im}\tau}\left(\left(Q+I^3\right)^2+\frac{k+2}{2k_i}\overline{P}_i^2\right)\right\rangle \\
&+\cdots ,
\end{aligned} \tag{2.9}$$

where the dots stand for higher orders in $\mathcal{F}$ and $\mathcal{R}$.

From now on we will assume that our ground state has at least $N=1$ supersymmetry[5]. In such ground states, terms in (2.9) that do not contain the helicity operator Q vanish because of the presence of the fermionic zero modes, and terms linear in Q vanish due to rotational invariance, $\langle I^3\rangle=0$. Thus for $N=1$ ground states, (2.9) becomes

$$D^{\text{string}}_{\text{one loop}}=\frac{8\pi^2(\operatorname{Im}\tau)^2}{k+2}\left\langle Q^2\left(\frac{\mathcal{R}^2}{k}\left(\left(\overline{I}^3\right)^2-\frac{k}{8\pi\operatorname{Im}\tau}\right)+\frac{\mathcal{F}^2}{2k_i}\left(\overline{P}_i^2-\frac{k_i}{4\pi\operatorname{Im}\tau}\right)\right)\right\rangle+\cdots . \tag{2.10}$$

The generic $N=1$ four-dimensional vacuum amplitude has the form

$$\langle 1\rangle=\frac{1}{\operatorname{Im}\tau\,|\eta|^4}\sum_{a,b=0,1}\frac{\vartheta\begin{bmatrix}a\\b\end{bmatrix}}{\eta}\,C\begin{bmatrix}a\\b\end{bmatrix}\Gamma\left(\frac{\mu}{M_s}\right)=0\,, \tag{2.11}$$

where $C\begin{bmatrix}a\\b\end{bmatrix}$ is the contribution of the internal conformal field theory, and

$$\Gamma(x)=-2x^2\frac{\partial}{\partial x}\Big[\sigma(x)-\sigma(2x)\Big]\ \text{with}\ \ \sigma(x)=\frac{1}{x}\sum_{m,n\in Z}\exp\left(-\frac{\pi}{\operatorname{Im}\tau\,x^2}|m+n\tau|^2\right) \tag{2.12}$$

at $x=(k+2)^{-\frac{1}{2}}=\mu/M_s$ stands for the $SO(3)_{\frac{k}{2}}$ partition function normalized so that $\lim_{x\to 0}\Gamma(x)=1$. This extra factor, which can be consistently removed, ensures the convergence of integrals such as those appearing in (2.7), at large values of $\operatorname{Im}\tau$. Expression (2.11) allows us to recast (2.10) as follows:

$$\begin{aligned}
D^{\text{string}}_{\text{one loop}}=-\frac{4\pi i}{k+2}\,\frac{\operatorname{Im}\tau}{|\eta|^4}\sum_{a,b=0,1}\Bigg\{&\frac{\mathcal{F}^2}{k_i}\,\frac{\partial_\tau\vartheta\begin{bmatrix}a\\b\end{bmatrix}}{\eta}\left(\overline{P}_i^2-\frac{k_i}{4\pi\operatorname{Im}\tau}\right)C\begin{bmatrix}a\\b\end{bmatrix}\Gamma\left(\frac{1}{\sqrt{k+2}}\right)\\
&-\frac{\mathcal{R}^2}{6k}\,\frac{\partial_\tau\vartheta\begin{bmatrix}a\\b\end{bmatrix}}{\eta}C\begin{bmatrix}a\\b\end{bmatrix}\left(\widehat{E}_2+\frac{2(k+2)}{i\pi}\partial_{\bar\tau}\right)\Gamma\left(\frac{1}{\sqrt{k+2}}\right)\Bigg\}+\cdots ,
\end{aligned} \tag{2.13}$$

where $\overline{P}_i^2$ acts as $\frac{i}{\pi}\frac{\partial}{\partial\bar\tau}$ on the appropriate subfactor of the 32 right-moving-fermion contribution, and

$$\widehat{E}_2\equiv\frac{6i}{\pi}\partial_{\bar\tau}\log\left(\operatorname{Im}\tau\,\bar\eta^2\right)=\overline{E}_2-\frac{3}{\pi\operatorname{Im}\tau}\,; \tag{2.14}$$

[5] The general formula in the absence of supersymmetry can be found in [12].

E_2 is an Eisenstein holomorphic function (see (3.7)) and $\widehat{E}_2$ is modular-covariant of degree 2. Since we are interested in the large-k limit, we can expand (2.13) in powers of $1/k$. In the next-to-leading order, the above expression reads:

$$D^{\text{string}}_{\text{one loop}} = -\frac{4\pi i}{k}\frac{\operatorname{Im}\tau}{|\eta|^4}\,\Gamma\left(\frac{1}{\sqrt{k+2}}\right)\sum_{a,b=0,1}\left\{\frac{\mathcal{F}^2}{k_i}\,\frac{\partial_\tau\vartheta\left[{a \atop b}\right]}{\eta}\left(\overline{P}^2_i - \frac{k_i}{4\pi\operatorname{Im}\tau}\right)C\begin{bmatrix}a\\b\end{bmatrix}\left(1-\frac{2}{k}\right)\right.$$
$$\left. - \frac{\mathcal{R}^2}{6k}\,\frac{\partial_\tau\vartheta\left[{a \atop b}\right]}{\eta}\widehat{E}_2\,C\begin{bmatrix}a\\b\end{bmatrix} + \mathcal{O}\left(\frac{1}{k^2}\right)\right\} + \cdots . \tag{2.15}$$

It deserves stressing here that the radiative corrections (2.15) include exactly the back-reaction of the gravitationally coupled fields; this accounts for the term $\propto \frac{1}{4\pi\operatorname{Im}\tau}$, which is universal and guarantees modular invariance.

In order to determine the string-induced one-loop renormalization for the gauge couplings, we have to compare the above free energy with the one that would have been computed in the low-energy field theory, in the presence of the same backgrounds. If one normalizes the effective theory generators such that the highest roots of the group algebra have length squared equal to 1 [6], one obtains (more details can be found in [14]):

$$\alpha'^2 F^{\text{effective}}_{\text{one loop}} = \frac{1}{k}\left(-Z_F\,\mathcal{F}^2 + \text{higher orders in } \mathcal{F} \text{ and } \mathcal{R}\right) + \mathcal{O}\left(\frac{1}{k^2}\right), \tag{2.16}$$

where Z_F stands for the vector multiplet wave-function renormalization, as appears in the low-energy effective action (see (A.1) and (A.2))

$$S^{\text{gauge sector}}_{\text{tree \& one loop}} = -\int d^4x\,\sqrt{G}\left(e^{-2\Phi} + Z_F\right)\sum_{i,a}\frac{1}{4g_i^2}\,F^a_{i\,\mu\nu}\,F^{a\,\mu\nu}_i . \tag{2.17}$$

Comparing eq. (2.16) with eqs. (2.7) and (2.15), and taking into account the difference between the corresponding normalizations, the net result for Z_F reads:

$$Z_F\left(\frac{\mu}{M_s}\right) = \frac{i}{16\pi^3 k_i}\int_{\mathcal{F}}\frac{d^2\tau}{\operatorname{Im}\tau}\,\frac{\Gamma(\mu/M_s)}{|\eta|^4}\sum_{a,b=0,1}\frac{\partial_\tau\vartheta\left[{a \atop b}\right]}{\eta}\left(\overline{P}^2_i - \frac{k_i}{4\pi\operatorname{Im}\tau}\right)C\begin{bmatrix}a\\b\end{bmatrix}. \tag{2.18}$$

Introducing as usual $g_{\text{string}} = \exp\langle\Phi\rangle$, we derive from eqs. (2.17) and (2.18) the effective one-loop string-corrected coupling $g_{i,\,\text{eff}}$:

$$\begin{aligned}\frac{16\,\pi^2}{g^2_{i,\,\text{eff}}} &= k_i\frac{16\,\pi^2}{g^2_{\text{string}}} + 16\,\pi^2 k_i\,Z_F\left(\frac{\mu}{M_s}\right)\\ &= k_i\frac{16\,\pi^2}{g^2_{\text{string}}} + \frac{i}{\pi}\int_{\mathcal{F}}\frac{d^2\tau}{\operatorname{Im}\tau}\,\frac{\Gamma(\mu/M_s)}{|\eta|^4}\sum_{a,b=0,1}\frac{\partial_\tau\vartheta\left[{a \atop b}\right]}{\eta}\left(\overline{P}^2_i - \frac{k_i}{4\pi\operatorname{Im}\tau}\right)C\begin{bmatrix}a\\b\end{bmatrix}.\end{aligned} \tag{2.19}$$

Equation (2.19) has been obtained by using an explicitly infra-red-regulated string loop amplitude. However, it is important to stress that the final relation between the running

[6]These are the usual normalizations that lead in particular to the tree-level relation $M_s = \frac{g_{\text{string}}}{\sqrt{32\,\pi G_N}}$.

gauge couplings of the low-energy field theory and the string coupling should not depend on how the infra-red has been regulated; put differently, this relation should not depend on the function $\Gamma(\frac{\mu}{M_s})$. In order to show this property, and eventually establish the expression for the running low-energy gauge couplings, we first isolate the contribution of the massless states responsible for the non-trivial infra-red behaviour of the integral in (2.18): we rewrite (2.19) in a form where we subtract and add back a $b_i \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau}\Gamma(\frac{\mu}{M_s})$ term, a manipulation perfectly well defined thanks to the presence of the regulator $\Gamma(\frac{\mu}{M_s})$. Here b_i are the full beta-function coefficients for the group factor G_i:

$$b_i = \lim_{\mathrm{Im}\,\tau\to\infty} \frac{i}{\pi}\frac{1}{|\eta|^4}\sum_{a,b=0,1}\frac{\partial_\tau\vartheta\begin{bmatrix}a\\b\end{bmatrix}}{\eta}\left(\overline{P}_i^2 - \frac{k_i}{4\pi\,\mathrm{Im}\,\tau}\right) C\begin{bmatrix}a\\b\end{bmatrix}. \tag{2.20}$$

Using the result

$$\int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau}\,\Gamma\left(\frac{\mu}{M_s}\right) = \log\frac{M_s^2}{\mu^2} + \log\frac{2e^{\gamma+3}}{\pi\sqrt{27}} + \mathcal{O}\left(\frac{\mu}{M_s}\right), \tag{2.21}$$

and taking the limit $\mu \to 0$ in the remaining integral of (2.19) since it does not suffer any longer from divergences at $\mathrm{Im}\,\tau \to \infty$ leads to:

$$\begin{aligned}\frac{16\,\pi^2}{g^2_{i,\,\mathrm{eff}}} &= k_i\frac{16\,\pi^2}{g^2_{\mathrm{string}}} + b_i\log\frac{M_s^2}{\mu^2} + b_i\log\frac{2\,e^{\gamma+3}}{\pi\sqrt{27}} \\ &\quad + \int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\,\tau}\left(\frac{i}{\pi}\frac{1}{|\eta|^4}\sum_{a,b=0,1}\frac{\partial_\tau\vartheta\begin{bmatrix}a\\b\end{bmatrix}}{\eta}\left(\overline{P}_i^2 - \frac{k_i}{4\pi\,\mathrm{Im}\,\tau}\right)C\begin{bmatrix}a\\b\end{bmatrix} - b_i\right).\end{aligned} \tag{2.22}$$

We can determine the running gauge couplings of the low-energy effective field theory by identifying the above string theory one-loop corrected coupling $g_{i,\,\mathrm{eff}}$ with the corresponding field theory one-loop gauge coupling, regulated in the infra-red in a similar fashion as the string theory. The effective field theory has also to be supplied with an ultraviolet cut-off. If we use dimensional regularization, we obtain the following field theory one-loop corrected coupling:

$$\left.\frac{16\,\pi^2}{g^2_{i,\,\mathrm{eff}}}\right|_{\text{field theory}} = \frac{16\,\pi^2}{g^2_{i,\,\mathrm{bare}}} + b_i\,(4\pi)^\epsilon\int_0^\infty\frac{dt}{t^{1-\epsilon}}\,\Gamma_{FT}\left(\frac{\mu}{\sqrt{\pi}M}\right), \tag{2.23}$$

where M is an arbitrary mass scale, and Γ_{FT} is the field theory counterpart of the string infra-red regulator, obtained by dropping all winding modes [7] in (2.12). On the other hand, one knows that in the $\overline{DR}$ scheme the relation between the field theory bare and running coupling is

$$\frac{16\,\pi^2}{g^2_{i,\,\mathrm{bare}}} = \frac{16\,\pi^2}{g_i^2(\mu)} - b_i\,(4\pi)^\epsilon\int_0^\infty\frac{dt}{t^{1-\epsilon}}\,e^{-t\frac{\mu^2}{M^2}}. \tag{2.24}$$

Plugging (2.24) into (2.23) and performing the resulting integral in the limit μ, $\epsilon \to 0$, leads to the following expression for the field theory one-loop corrected coupling:

$$\left.\frac{16\,\pi^2}{g^2_{i,\,\mathrm{eff}}}\right|^{\overline{DR}}_{\text{field theory}} = \frac{16\,\pi^2}{g_i^2(\mu)} + b_i\,(2\gamma+2); \tag{2.25}$$

[7] The extra $\sqrt{\pi}$ in the argument of Γ_{FT} accounts for the identification of the (dimensionless in the above convention) Schwinger proper-time parameter t with $\pi\,\mathrm{Im}\,\tau$.

identifying the latter with (2.22), we finally obtain [13] the already anticipated eq. (1.1) with

$$\Delta_i = \int_{\mathcal{F}} \frac{d^2\tau}{\operatorname{Im}\tau} \left(\frac{i}{\pi} \frac{1}{|\eta|^4} \sum_{a,b=0,1} \frac{\partial_\tau \vartheta\begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \left(\overline{P}_i^2 - \frac{k_i}{4\pi \operatorname{Im}\tau} \right) C\begin{bmatrix} a \\ b \end{bmatrix} - b_i \right) + b_i \log \frac{2\, e^{1-\gamma}}{\pi\sqrt{27}} \,. \qquad (2.26)$$

These are the full threshold corrections in the $\overline{DR}$ scheme. As advertised previously, this expression no longer depends on the infra-red regularization prescription. This result could have been anticipated as a consequence of the cancellation of the infra-red divergences between the fundamental and the effective theory since they have the same massless spectrum. However it could only be proved [13] in the presence of a consistent infra-red regulator, similar in both theories. Moreover, it is important to emphasize that (2.26) contains rigorously all universal terms that were missing in previous approaches [3, 4] and that we will be analysing in the sequel.

We would like to stress that the computation we presented here was performed in the *dilaton frame* for *both* string theory and the effective field theory in which we have used a moduli-dependent convention for the masses. To put it differently, the kinetic terms of the various fields are normalized to one, and this is a physical basis. In this frame, the string perturbative expansion appears as a power series with respect to the coupling $g_{\text{string}} = \exp\langle\Phi\rangle$, which is a well defined parameter that remains unaltered at any order of the expansion. Therefore, it provides an unambiguous control of the latter. Moreover, as long as the string ground state possesses at least $N = 1$ supersymmetry, the Planck scale M_P does not receive any correction in perturbation theory [14], which means that the tree-level relation (1.2) holds to all orders. This property plays an important role in the low-energy unification of the effective couplings [13].

Instead of the dilaton frame, one could use the S-frame in the effective supergravity. In that case, the expansion parameter is $\frac{1}{\operatorname{Im} S}$, and this turns out to be convenient for analysing the holomorphicity and duality properties which are somehow obscured in the dilaton frame, and consequently the issue of supersymmetry. On the other hand, this parameter must be redefined at each order of the perturbative expansion, as a consequence of the antisymmetric-moduli mixing due to the Green–Schwarz term [5]. This term changes the definition of the axion and, by supersymmetry, that of the dilaton. Thus, terms of order n in the S-frame get contributions from all loop orders up to the nth. Furthermore, a new universal-threshold-like correction appears now along with the thresholds (2.26), which is again a remnant of the ten-dimensional Green–Schwarz term [5, 6]. This term is only present in $N = 1$ models and is moduli-dependent in contrast (see section 4) with the $(N = 1)$-sector contribution of the corrections (2.26). This extra moduli dependence, which enters from the effective field theory matching in the S-frame, is responsible for the modification of the analytic properties in this frame. The appearance of the Green–Schwarz term in the S-frame is summarized in appendix A.

As a final remark, we would like to comment on the structure of the threshold corrections as they appear in the dilaton frame, which is the frame that we will be using in our subsequent computations. Part of the thresholds (2.26) is universal and this enables us to split (2.26) according to eq. (1.3). As we have already mentioned, the universal piece Y in (1.3) contains,

among other things, contributions from the universal sector (gravity in particular). Such contributions are not taken into account in grand unified theories while in string theory they are well defined and calculable quantities. Moreover, Y is infra-red-finite, which in particular means that it is continuous and remains finite when extra states become massless at some special values of the moduli. Thus Y is a finite correction to the "bare" string coupling $g_{\rm string}$, and we can write (1.1) as

$$\frac{16\,\pi^2}{g_i^2(\mu)} = k_i \frac{16\,\pi^2}{g^2_{\rm renorm}} + b_i \log \frac{M_s^2}{\mu^2} + \hat{\Delta}_i\,, \tag{2.27}$$

where we have defined a "renormalized" string coupling by [13]

$$g^2_{\rm renorm} = \frac{g^2_{\rm string}}{1 - \frac{Y}{16\,\pi^2}\, g^2_{\rm string}}\,. \tag{2.28}$$

Of course, such a coupling is meaningful provided it appears as the natural expansion parameter in several amplitudes that are relevant for the low-energy string physics. In general, this might not be the case as a consequence of some arbitrariness in the decomposition (1.3). Examples of this kind arise in $N = 1$ models (see the Z_4 orbifold in section 4) as well as in certain $N = 2$ constructions [24]. It is important to keep in mind that this "renormalized" string coupling is defined here in a *moduli-dependent* way. This moduli dependence affects the string unification [13]. Indeed, as we will see in the sequel, when proper unification of the couplings appears, namely when $\hat{\Delta}_i$ can be written as $b_i \Delta$, their common value at the unification scale is $g_{\rm renorm}$, which plays therefore the role of a phenomenological parameter. Moreover, the unification scale turns out to be proportional to M_s. The latter can be expressed in terms of the "low-energy" parameters $g_{\rm renorm}$ and M_P, by using eq. (1.2) and its non-renormalization property [14], as well as (2.28):

$$M_s = \frac{M_P\, g_{\rm renorm}}{\sqrt{1 + \frac{Y}{16\,\pi^2}\, g^2_{\rm renorm}}}\,; \tag{2.29}$$

this involves the moduli-dependent function Y. As is shown in appendix A, relation (2.29) holds also in the S-frame where, at one loop, $g^{-2}_{\rm renorm} = {\rm Im}\, S + \frac{\Delta^{GS}}{16\,\pi^2}$ with Δ^{GS} the Green–Schwarz term (see eq. (A.18)). Despite the presence of the moduli-dependent universal function Δ^{GS}, the string scale M_s, and consequently the unification scale, are only affected by the universal threshold Y.

3. Universal thresholds for a class of $N = 2$ theories

Let us now concentrate on $N = 2$ ground states. We will focus on models that come from toroidal compactification of generic six-dimensional $N = 1$ string theories. There are of course more general $N = 2$ models in four dimensions that cannot be viewed as toroidal compactifications of a six-dimensional theory [24]. These will be dealt with in detail in a separate publication. In the cases at hand, however, there is a universal two-torus, which

provides the (perturbative) central charges of the $N = 2$ algebra. Therefore, (2.26) becomes

$$\Delta_i = \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau} \left(\frac{\Gamma_{2,2}\left(T, U, \overline{T}, \overline{U}\right)}{\bar{\eta}^{24}} \left(\overline{P}_i^2 - \frac{k_i}{4\pi\,\mathrm{Im}\,\tau} \right) \overline{\Omega} - b_i \right) + b_i \log \frac{2\, e^{1-\gamma}}{\pi\sqrt{27}} \,, \tag{3.1}$$

where T and U are the complex moduli of the two-torus, $\overline{\Omega}$ is an antiholomorphic function and

$$\Gamma_{2,2}\left(T, U, \overline{T}, \overline{U}\right) = \sum_{\mathbf{m},\mathbf{n} \in Z} \exp\left(2\pi i \tau \left(m_1\, n^1 + m_2\, n^2 \right) \right.$$
$$\left. - \frac{\pi\,\mathrm{Im}\,\tau}{\mathrm{Im}\,T\,\mathrm{Im}\,U} \left| T n^1 + T U n^2 + U m_1 - m_2 \right|^2 \right) . \tag{3.2}$$

From (3.1), we observe that the function

$$\overline{F}_i = \frac{1}{\bar{\eta}^{24}} \left(\overline{P}_i^2 - \frac{k_i}{4\pi\,\mathrm{Im}\,\tau} \right) \overline{\Omega} \tag{3.3}$$

is modular invariant. Consider the associated function that appears in the R^2-term renormalization (see eq. (2.15) or ref. [9] for more details),

$$\overline{F}_{\mathrm{grav}} = \frac{\widehat{\overline{E}}_2}{12} \frac{\overline{\Omega}}{\bar{\eta}^{24}} = \frac{1}{\bar{\eta}^{24}} \left(\frac{i}{\pi} \partial_{\bar{\tau}} \log \bar{\eta} - \frac{1}{4\pi\,\mathrm{Im}\,\tau} \right) \overline{\Omega} \,, \tag{3.4}$$

which is also modular invariant, and eventually leads to the gravitational anomaly. The difference $\overline{F}_i - k_i\, \overline{F}_{\mathrm{grav}}$ is an antiholomorphic function, which is modular invariant. It has at most a simple pole at $\tau \to i\infty$ (associated with the heterotic unphysical tachyon) and is finite inside the moduli space of the torus. This implies that

$$\overline{F}_i = k_i\, \overline{F}_{\mathrm{grav}} + A_i\, \bar{j}(\bar{\tau}) + B_i \,, \tag{3.5}$$

where A_i and B_i are constants to be determined, and $j(\tau) = \frac{1}{q} + 744 + \mathcal{O}(q)$, $q = \exp(2\pi i \tau)$ is the standard j-function. The modular invariance of $\overline{F}_{\mathrm{grav}}$ implies that $\overline{\Omega}$ is a modular form of weight 10, which is finite inside the moduli space. This property fixes

$$\overline{\Omega} = \xi\, \overline{E}_4\, \overline{E}_6 \,, \tag{3.6}$$

where E_{2n} is the nth Eisenstein series:

$$E_2 = \frac{12}{i\pi} \partial_\tau \log \eta = 1 - 24 \sum_{n=1}^{\infty} \frac{n\, q^n}{1 - q^n} \,, \tag{3.7}$$

$$E_4 = \frac{1}{2} \left(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8 \right) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} \,, \tag{3.8}$$

$$E_6 = \frac{1}{2} \left(\vartheta_2^4 + \vartheta_3^4 \right) \left(\vartheta_3^4 + \vartheta_4^4 \right) \left(\vartheta_4^4 - \vartheta_2^4 \right) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} \,. \tag{3.9}$$

Putting everything together in (3.1) we obtain:

$$\Delta_i = \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau} \left(\Gamma_{2,2}\left(T,U,\overline{T},\overline{U}\right)\left(\frac{\xi k_i}{12}\frac{\widehat{E}_2\,\overline{E}_4\,\overline{E}_6}{\bar{\eta}^{24}} + A_i\,\bar{j} + B_i\right) - b_i\right) + b_i \log\frac{2\,e^{1-\gamma}}{\pi\sqrt{27}}\,. \quad (3.10)$$

There are two constraints that allow us to fix the constants A_i, B_i. The first is that the $1/\bar{q}$ pole is absent from the group trace, which gives

$$A_i = -\frac{\xi k_i}{12}\,. \quad (3.11)$$

The second is (2.20), which implies

$$744\,A_i + B_i - b_i + k_i\,b_{\mathrm{grav}} = 0\,, \quad (3.12)$$

where the constant term in the large- $\mathrm{Im}\,\tau$ expansion of $\overline{F}_{\mathrm{grav}}$

$$b_{\mathrm{grav}} = \lim_{\mathrm{Im}\,\tau\to\infty}\left(\overline{F}_{\mathrm{grav}} - \frac{\xi}{12}\frac{1}{\bar{q}}\right) = -22\,\xi \quad (3.13)$$

is the gravitational anomaly in units where a hypermultiplet contributes $\frac{1}{12}$ [9]. Plugging (3.11)–(3.13) in (3.10), we finally obtain:

$$\begin{aligned}\Delta_i \;&=\; b_i\left(\log\frac{2\,e^{1-\gamma}}{\pi\sqrt{27}} + \int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\,\tau}\left(\Gamma_{2,2}\left(T,U,\overline{T},\overline{U}\right) - 1\right)\right)\\ &\quad + \frac{\xi k_i}{12}\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\,\tau}\,\Gamma_{2,2}\left(T,U,\overline{T},\overline{U}\right)\left(\frac{\widehat{E}_2\,\overline{E}_4\,\overline{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008\right)\,. \end{aligned} \quad (3.14)$$

The first integral in (3.14) was computed explicitly in [4] and recently generalized in [17]:

$$\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\,\tau}\left(\Gamma_{2,2}\left(T,U,\overline{T},\overline{U}\right) - 1\right) = -\log\left(|\eta(T)|^4|\eta(U)|^4\,\mathrm{Im}\,T\,\mathrm{Im}\,U\right) - \log\frac{8\pi\,e^{1-\gamma}}{\sqrt{27}}\,. \quad (3.15)$$

Therefore, as advertised above, we can write[8]

$$\Delta_i = b_i\,\Delta - k_i\,Y\,, \quad (3.16)$$

with

$$\Delta = -\log\left(4\pi^2|\eta(T)|^4|\eta(U)|^4\,\mathrm{Im}\,T\,\mathrm{Im}\,U\right) \quad (3.17)$$

and

$$Y = -\frac{\xi}{12}\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\,\tau}\,\Gamma_{2,2}\left(T,U,\overline{T},\overline{U}\right)\left(\left(\overline{E}_2 - \frac{3}{\pi\,\mathrm{Im}\,\tau}\right)\frac{\overline{E}_4\,\overline{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008\right) \quad (3.18)$$

(we have used eq. (2.14)). This form of the universal term was determined for the case of $Z_2\times Z_2$ orbifolds in [13] and further discussed in [14].

[8]This is not true for more general $N=2$ ground states (see [24]).

The coefficient ξ can be related to the number of massless vector multiplets N_V and hypermultiplets N_H via the gravitational anomaly ($b_{\rm grav}$), which can also be computed from the low-energy theory of massless states. In units where a scalar contributes 1, the graviton contributes 212, the antisymmetric tensor 91, the gravitino $-\frac{233}{4}$, a vector -13 and a Majorana fermion $\frac{7}{4}$; therefore the $N=2$ supergravity multiplet contributes $212-2\frac{233}{4}-13=\frac{165}{2}$, the tensor multiplet contributes $-13+2\frac{7}{4}+1+91=\frac{165}{2}$, a vector multiplet $-13+2\frac{7}{4}+2=-\frac{15}{2}$ and a hypermultiplet $2\frac{7}{4}+4=\frac{15}{2}$. Thus in the units of (3.13),

$$b_{\rm grav}=\frac{22-N_V+N_H}{12}\,, \tag{3.19}$$

and hence

$$\xi=-\frac{1}{264}\left(22-N_V+N_H\right)\,. \tag{3.20}$$

For the models at hand we can go even further and completely determine ξ. One can indeed show that N_H-N_V is a *universal constant* for the whole class of four-dimensional $N=2$ models obtained by toroidal compactification of *any* $N=1$ ground state in six dimensions. The argument is the following. From the six-dimensional point of view, the models at hand must obey an anomaly-cancellation constraint, which reads: $N_H-N_V|_{\rm six\ dim}=244$, and does not depend on the kind of compactification that has been performed from ten to six dimensions[9] [35]. After two-torus compactification to four dimensions, two extra $U(1)$'s appear, leading to the relation

$$N_H-N_V=242 \tag{3.21}$$

between the numbers of vector multiplets and hypermultiplets. In turn, eq. (3.20) implies that for this class of ground states $\xi=-1$. As a consequence, all $N=2$ models under consideration have equal universal thresholds, given by

$$Y\left(T,U,\overline{T},\overline{U}\right)=\frac{1}{12}\int_{\mathcal F}\frac{d^2\tau}{{\rm Im}\,\tau}\,\Gamma_{2,2}\left(T,U,\overline{T},\overline{U}\right)\left(\left(\overline{E}_2-\frac{3}{\pi\,{\rm Im}\,\tau}\right)\frac{\overline{E}_4\,\overline{E}_6}{\overline{\eta}^{24}}-\overline{j}+1008\right). \tag{3.22}$$

As an example, consider the case of the Z_2 orbifold, where we have a gauge group $E_8\times E_7\times SU(2)\times U(1)^2$ and thus $N_V=386$. The number of massless hypermultiplets is $N_H=628$. Using these numbers in (3.20) we obtain indeed $\xi=-1$. As expected by supersymmetry, the corresponding universal threshold (3.22) is twice as big as a single-plane contribution of the symmetric $Z_2\times Z_2$ orbifold analysed in [13][10].

Expression (3.22) can be further simplified if one uses a generalization of (3.15), valid for more general modular-invariant functions, to integrate the last terms:

$$Y\left(T,U,\overline{T},\overline{U}\right)\;=\;\frac{1}{12}\int_{\mathcal F}\frac{d^2\tau}{{\rm Im}\,\tau}\,\Gamma_{2,2}\left(T,U,\overline{T},\overline{U}\right)\left(\left(\overline{E}_2-\frac{3}{\pi\,{\rm Im}\,\tau}\right)\frac{\overline{E}_4\,\overline{E}_6}{\overline{\eta}^{24}}+264\right)$$

[9]Actually, this constraint, which ensures that ${\rm Tr}\,R^4$ vanishes, holds even when there occurs a symmetry enhancement originated from non-perturbative effects, provided the number of tensor multiplets remains $N_T=1$. Note that this six-dimensional anomaly-cancellation constraint is also used in [34], in relation to four-dimensional quantities.

[10]The universal threshold computed in [13] corresponds to the three-plane contribution of the $Z_2\times Z_2$ model with $T_i=T$ and $U_i=U$ $\forall i$. A factor 2/3 is therefore needed to recover (3.22).

$$+\frac{1}{3}\log|j(T)-j(U)|\,. \tag{3.23}$$

Using the method of orbits of the modular group, the remaining integral in (3.23) can also be reduced to a multiple series expansion (see [17]). This is detailed in appendix B, where we use a product representation of $j(T)-j(U)$ to make more transparent the cancellation of the logarithmic divergences occurring in both terms of (3.23) when $T \to U$ as a consequence of the appearance of extra massless states. It is actually shown that Y is *finite and continuous* inside the whole moduli space. We also present in this appendix numerical evaluation and plots as well as various large-moduli behaviours of (3.22); the latter have some relevance in the context of the decompactification problem (see [24]).

As far as the issue of unification is concerned, several observations are in order. Although $N=2$ models are not directly relevant for phenomenology, it is nevertheless interesting to note that universal thresholds always decrease the unification scale. Indeed, by using eqs. (2.27) and (3.16), it appears that unification of all couplings takes place (in the $\overline{DR}$ scheme) at a scale

$$M_U = M_P\, g_U\, e^{\frac{\Delta}{2}}\,\frac{1}{\sqrt{1+\frac{Y}{16\,\pi^2}\,g_U^2}}\,, \tag{3.24}$$

where

$$g_U \equiv g_\alpha(M_U) = \frac{g_{\rm string}}{\sqrt{1-\frac{Y}{16\,\pi^2}\,g_{\rm string}^2}} \tag{3.25}$$

for any group factor (this is the renormalized coupling introduced in (2.28)), Δ and Y are given by eqs. (3.17) and (3.23) respectively, and we have used explicitly (1.2) in order to express the unification scale in terms of the effective field theory parameters M_P and g_U. The last factor in (3.24) is due to the existence of the universal terms which lead to a shift of the dilaton field in order to reabsorb the universal contributions into the string coupling. It is interesting to observe that[11] $Y(R_1, R_2) > 0$ (see appendix B). Therefore, this extra factor always gives a lower unification scale with respect to the case where these terms are neglected. On the other hand the first factor in (3.24) monotonically increases for radii moving away from the self-dual point, while the second one monotonically decreases. Following [13] we conclude that the minimum unification scale is reached at the self-dual point $R_1 = R_2 = 1$ with the value

$$M_U^{\rm min} \approx 5.56\times 10^{17}\times g_U \times \frac{1}{\sqrt{1+0.15\times g_U^2}}\;{\rm GeV}\,. \tag{3.26}$$

The last factor in this expansion represents the effect of the universal thresholds. Note that results (3.24) and (3.26) are valid for the whole class of $N=2$ models that were analysed here above.

Besides the relevance that the universal contributions Y might have in the framework of string unification, we should mention that they are also related to the one-loop correction of the Kähler potential for the moduli fields [5, 6, 7, 15] (see also appendix A, eqs. (A.7) and

[11] We consider here the case $\mathrm{Re}\,T = \mathrm{Re}\,U = 0$, and we parametrize it as usually: $\mathrm{Im}\,T = R_1\,R_2$ and $\mathrm{Im}\,U = R_2/R_1$.

(A.15)), as is expected from supersymmetry:

$$K^{(1)}\left(T,U,\overline{T},\overline{U}\right) = -\frac{1}{16\,\pi^2}\,Y\left(T,U,\overline{T},\overline{U}\right) + \kappa\Big(T,U\Big) + \bar{\kappa}\left(\overline{T},\overline{U}\right)\,; \tag{3.27}$$

here $\kappa(T,U)$ is an analytic function which is irrelevant in the determination of the Kähler metric but plays a role in the duality covariance of $K^{(1)}$. By using the identity

$$\frac{1}{(\,\mathrm{Im}\,T)^2}\,\frac{\partial^2}{\partial\tau\partial\bar{\tau}}\,(\,\mathrm{Im}\,\tau\,\Gamma_{2,2}) = \frac{1}{\mathrm{Im}\,\tau}\,\frac{\partial^2}{\partial T\partial\overline{T}}\,\Gamma_{2,2} \tag{3.28}$$

in eqs. (3.22) and (3.27), it is easy to show that

$$\begin{aligned}\frac{\partial^2 K^{(1)}}{\partial T\partial\overline{T}} \quad = \quad & -\frac{1}{128\,\pi^2\,(\,\mathrm{Im}\,T)^2}\int_{\mathcal{F}}\frac{d^2\tau}{(\,\mathrm{Im}\,\tau)^2}\,\frac{i}{\pi}\,\frac{\partial}{\partial\bar{\tau}}(\,\mathrm{Im}\,\tau\,\Gamma_{2,2})\,\frac{\overline{E}_4\,\overline{E}_6}{\bar{\eta}^{24}}\\ & -\frac{1}{192\,\pi^2\,(\,\mathrm{Im}\,T)^2}\int_{\mathcal{F}} d^2\tau\,\frac{\partial}{\partial\tau}\,G(\tau,\bar{\tau})\,,\end{aligned} \tag{3.29}$$

where

$$G(\tau,\bar{\tau}) = \left(\frac{i}{2}\,\Gamma_{2,2} + \,\mathrm{Im}\,\tau\frac{\partial}{\partial\bar{\tau}}\Gamma_{2,2}\right)\left(\frac{\overline{E}_2\,\overline{E}_4\,\overline{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008 - \frac{3}{\pi\,\mathrm{Im}\,\tau}\,\frac{\overline{E}_4\,\overline{E}_6}{\bar{\eta}^{24}}\right)\,. \tag{3.30}$$

The first term in (3.29) is indeed the one-loop correction to the Kähler metric $K^{(1)}_{T\overline{T}}$ as it appears in [8][12] or [15]. The second one is a boundary term that vanishes for generic values of the moduli. However, when $T\sim U$, $\Gamma_{2,2}\sim 1+\bar{q}+\cdots$, and this term might develop finite or even δ-function contributions. Actually, δ-functions are originated around $T=U$ as

$$\begin{aligned}\lim_{\mathrm{Im}\,\tau\to\infty}\,\mathrm{Im}\,\tau\left(\frac{\partial}{\partial\bar{\tau}}\Gamma_{2,2}\right)\frac{1}{\bar{q}} \quad &= \quad -4\pi i\lim_{\mathrm{Im}\,\tau\to\infty}\,\mathrm{Im}\,\tau\,\exp\left(-\pi\,\mathrm{Im}\,\tau\frac{|T-U|^2}{2\,\mathrm{Im}\,T\,\mathrm{Im}\,U}\right)\\ &= \quad -8\pi i\,(\,\mathrm{Im}\,T)^2\,\delta^2(T-U)\,.\end{aligned} \tag{3.31}$$

It is nice to observe that these contributions are avoided for the same reason that Y remains free of infra-red divergences when the gauge symmetry gets enlarged: the absence of poles in $\frac{\overline{E}_2\,\overline{E}_4\,\overline{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008$. However, the term $-\frac{3}{\pi\,\mathrm{Im}\,\tau}\,\frac{\overline{E}_4\,\overline{E}_6}{\bar{\eta}^{24}}$ in (3.30) will generate finite boundary contributions at $T=U$, which will be further enhanced when $T=U=i$ and $T=U=\rho=\exp\frac{2\pi i}{3}$. More precisely we have

$$-\frac{1}{192\,\pi^2\,(\,\mathrm{Im}\,T)^2}\int_{\mathcal{F}} d^2\tau\,\frac{\partial}{\partial\tau}\,G(\tau,\bar{\tau}) = -\frac{\lambda}{64\,\pi^2\,(\,\mathrm{Im}\,T)^2} \tag{3.32}$$

with $\lambda=6$ for $T=U=\rho$, $\lambda=4$ for $T=U=i$, $\lambda=2$ for generic $T=U$, and $\lambda=0$ elsewhere.

[12]To make contact with this reference, one can use the following identity:

$$\frac{E_4\,E_6}{\eta^{24}} = \big(j(i)-j(\tau)\big)\left(\frac{\partial\log j}{\partial\log q}\right)^{-1}\,.$$

Finally, by using expression (B.11) it is easy to show that Y is continuous at $T = U$ as long as the moduli remain finite. On the other hand, $\partial_T \partial_{\overline{T}} Y$ develops logarithmic singularities:

$$\partial_T \partial_{\overline{T}} Y \sim -\frac{1}{(\operatorname{Im} T)^2} \log |T - U| + \text{ regular terms,} \tag{3.33}$$

around $T \sim U$, as is expected from general arguments. Those singularities are responsible for non-trivial monodromy properties of the prepotential in these $N = 2$ models [8, 11, 17]. In connection with these properties, a comment is in order here: our present treatment implies that the results of [17] for the case of vanishing Wilson lines provide the prepotential for all $N = 2$ models that are toroidal compactifications of $N = 1$ six-dimensional theories. Furthermore, this holds also for the special classes of Wilson lines dealt with in [17].

4. The case of $N = 1$ orbifolds

We come now to the $N = 1$ heterotic compactifications on orbifolds $\mathbf{T}^6/G$. We will restrict to the case where G is Abelian. Although there is not any fundamental obstruction with the non-Abelian situation, the computation is expected to be more complicated. In a generic $N = 1$ orbifold compactification, both $N = 1$ and $N = 2$ supersymmetric sectors contribute to the gauge coupling renormalization. This allows us to express the thresholds (2.26), which appear in eq. (1.1), as

$$\begin{aligned} \Delta_i &= \Delta_i^{N=1} + \Delta_i^{N=2} \\ &= \sum_{N=1,2} \int_{\mathcal{F}} \frac{d^2\tau}{\operatorname{Im}\tau} \left(\frac{i}{\pi} \frac{1}{|\eta|^4} \sum_{a,b=0,1} \frac{\partial_\tau \vartheta\left[{a \atop b}\right]}{\eta} \left(\overline{P}_i^2 - \frac{k_i}{4\pi \operatorname{Im}\tau} \right) C^N \begin{bmatrix} a \\ b \end{bmatrix} - b_i^N \right) \\ &\quad + b_i^N \log \frac{2\, e^{1-\gamma}}{\pi\sqrt{27}} , \end{aligned} \tag{4.1}$$

where $b_i^{N=1,2}$ are the contributions originated from $N = 1, 2$ supersymmetric sectors and $b_i = b_i^{N=1} + b_i^{N=2}$ are the full beta-function coefficients of the $N = 1$ model.

Let us focus for the moment on the $(N = 2)$-sector contributions, which exhaust all the dependence of the thresholds on the untwisted moduli of the torus $\mathbf{T}^6/G$. Such sectors are present when some twists $g \in G$ have unit eigenvalues, leaving therefore unrotated a complex plane of $\mathbf{T}^6$. As was described in [4], these twists form a disjoint union $\bigcup_\alpha G_\alpha \subset G$: they only share the identity corresponding to the $N = 4$ sector. Each G_α is the little group of a given complex plane of $\mathbf{T}^6$, which needs not be the same for all G_α's. Furthermore, the subset of sectors generated by twists $g \in G_\alpha$ appears actually as the set of all twisted sectors that define an orbifold model on $\mathbf{T}^6/G_\alpha$. Thus we conclude that

$$\Delta_i^{N=2}\left[\mathbf{T}^6/G\right] = \sum_\alpha \frac{|G_\alpha|}{|G|} \Delta_i \left[\mathbf{T}^6/G_\alpha\right] . \tag{4.2}$$

On the other hand, the orbifold model on $\mathbf{T}^6/G_\alpha$ has $N = 2$ supersymmetry and can be viewed as a two-torus compactification of a $N = 1$ supersymmetric model in six dimensions,

the two-torus being the complex plane of $\mathbf{T}^6$ left invariant under G_α. Hence it belongs to the class of models that have been studied in section 3, leading automatically to the following result:

$$\Delta_i^{N=2} = -\sum_\alpha b_i^\alpha \frac{|G_\alpha|}{|G|} \log\left(4\pi^2 |\eta(T_\alpha)|^4 |\eta(U_\alpha)|^4 \operatorname{Im} T_\alpha \operatorname{Im} U_\alpha\right) - k_i\, Y^{N=2} \tag{4.3}$$

with

$$Y^{N=2} = \sum_\alpha \frac{|G_\alpha|}{|G|}\, Y\left(T_\alpha, U_\alpha, \overline{T}_\alpha, \overline{U}_\alpha\right), \tag{4.4}$$

where $Y\left(T_\alpha, U_\alpha, \overline{T}_\alpha, \overline{U}_\alpha\right)$ is given by eq. (3.22). In eqs. (4.3) and (4.4), T_α and U_α are the moduli corresponding to the complex plane whose little group is G_α, and b_i^α are the beta-function coefficients of the $N=2$ orbifold $\mathbf{T}^6/G_\alpha$, which allow us to express the $N=2$ contributions to the beta-function coefficients as follows:

$$b_i^{N=2} = \sum_\alpha b_i^\alpha \frac{|G_\alpha|}{|G|}\,. \tag{4.5}$$

A few remarks are in order here. At the level of the $\mathbf{T}^6/G_\alpha$ orbifold, the moduli T_α and U_α are completely free. When brought down to the $N=1$ model, however, some of them might be constrained in order to fit the discrete symmetry that is modded out. Therefore, some of the arguments in expressions (4.3) and (4.4) are in general frozen to some specific values (more details about that can be found in [4]).

As far as stringy $N=1$ contributions are concerned, no moduli dependence appears in the thresholds since none of the corresponding twists acts trivially on any plane inside $\mathbf{T}^6$. On the other hand, no systematic factorization of a real function such as $\Gamma_{2,2}$ is possible in the integrand of the first term of (4.1). Consequently, one cannot advocate holomorphicity and analytic properties to determine the generic structure for this threshold, as was done in section 3 (see eqs. (3.1) and (3.6)). Nevertheless, in order to get a better appreciation of its influence on the low-energy physics, we can proceed to a numerical estimation in the case of two models, namely the symmetric Z_3 and Z_4 orbifolds. Our starting point is eq. (4.1), that we can recast in the following form:

$$\Delta_i^{N=1} = b_i^{N=1} \log \frac{2\, e^{1-\gamma}}{\pi\sqrt{27}} + \delta_i\,, \tag{4.6}$$

where

$$\delta_i = \int_{\mathcal{F}} \frac{d^2\tau}{\operatorname{Im}\tau}\left(d_i\;(\tau,\bar\tau) - b_i^{N=1}\right) - \frac{k_i}{4\pi}\int_{\mathcal{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2}\, y(\tau,\bar\tau) \tag{4.7}$$

and $d_i(\tau,\bar\tau)$ and $y(\tau,\bar\tau)$ are model-dependent functions.

(i) The symmetric Z_3 orbifold

In this model, the gauge group is $E_8\times E_6\times SU(3)$ and the absence of $N=2$ sectors implies that $\Delta_i^{N=2} = b_i^{N=2} = 0$. The beta-function coefficients are ($b_i \equiv b_i^{N=1}$) $b_{E_8} = -90$, $b_{E_6} = 72$ and $b_{SU(3)} = 72$ respectively, while the functions introduced here above read ($q = e^{2\pi i\tau}$):

$$d_{E_8} = -90 - 540\, q + 26460\, \bar q + 918540\, q^{\frac{1}{3}}\, \bar q^{\frac{4}{3}} + 158760\, q\, \bar q + 2963520\, \bar q^2 + \mathcal{O}\left(q^{r>2}\right),$$

$$
\begin{aligned}
d_{E_6} &= 72 + 2916\, q^{\frac{1}{3}}\, \bar{q}^{\frac{1}{3}} + 432\, q + 27432\, \bar{q} + 2916\, q^{\frac{4}{3}}\, \bar{q}^{\frac{1}{3}} + 921456\, q^{\frac{1}{3}}\, \bar{q}^{\frac{4}{3}} \\
&\quad + 164592\, q\, \bar{q} + 2963520\, \bar{q}^2 + \mathcal{O}\left(q^{r>2}\right) , \\
d_{SU(3)} &= d_{E_6} , \\
y &= 108 - \frac{3}{2}\frac{1}{\bar{q}} - 9\,\frac{q}{\bar{q}} + 15309\, q^{\frac{1}{3}}\, \bar{q}^{\frac{1}{3}} + 648\, q + 150174\, \bar{q} + 15309\, q^{\frac{4}{3}}\, \bar{q}^{\frac{1}{3}} \\
&\quad + 4496472\, q^{\frac{1}{3}}\, \bar{q}^{\frac{4}{3}} - 9\,\frac{q^3}{\bar{q}} + 9010444\, q\, \bar{q} + 12070176\, \bar{q}^2 + \mathcal{O}\left(q^{r>2}\right) .
\end{aligned}
\tag{4.8}
$$

After numerical integration, taking into account $\mathcal{O}\left(q^{\frac{7}{2}}\right)$ terms, we obtain:

$$
\delta_{E_8} \approx -16.0 \; , \quad \delta_{E_6} = \delta_{SU(3)} \approx -5.0 \, . \tag{4.9}
$$

Note that $\delta_{SU(3)} - \delta_{E_8} \approx 11.0$, in agreement with [3]. In this case , due to the fact that $d_{E_6} = d_{SU(3)}$ we can express the thresholds as for the $N=2$ models (eq. (3.16)):

$$
\Delta_i^{N=1} = b_i^{N=1}\, \Delta^{N=1} - Y^{N=1} , \tag{4.10}
$$

where

$$
\Delta^{N=1} = \log \frac{2\, e^{1-\gamma}}{\pi\sqrt{27}} + \delta \, . \tag{4.11}
$$

Our numerical results lead to $\delta \approx 0.07$ and $Y^{N=1} \approx 9.82$. However, this is not to be considered as a generic feature of $N=1$ sectors, and we will meet a counter example bellow. In the case under consideration, using eqs. (1.1) and (4.10) as well as the non-renormalization theorem [14] for (1.2), we can define a common unification scale for all gauge couplings:

$$
M_U = M_P\, g_U \sqrt{\frac{2\, e^{1-\gamma+\delta}}{\pi\sqrt{27}}}\; \frac{1}{\sqrt{1 + \frac{Y^{N=1}}{16\,\pi^2}\, g_U^2}} , \tag{4.12}
$$

where we introduced, as previously, the effective field theory parameters M_P and $g_U = \frac{1}{g_i(M_U)}$ which is the common coupling at the unification scale. This coupling is again the one that was introduced in section 2 as the "renormalized" string coupling, g_{renorm} (see eqs. (2.27) and (2.28)). Contrary to the $N=2$ case, here it is not possible to shift M_U by moving the moduli. Using our numerical results, we obtain:

$$
M_U \approx 5.4 \times 10^{17} \times g_U \times \frac{1}{\sqrt{1 + 0.06 \times g_U^2}} \, \text{GeV}. \tag{4.13}
$$

Thus the universal contributions lead to a small decrease of the unification scale, which is of the order of 3% for $g_U \sim 1$.

(ii) The symmetric Z_4 orbifold

The gauge group is now $E_8 \times E_6 \times SU(2) \times U(1)$. Here the set of $N=2$ sectors is that of the symmetric Z_2 orbifold and the corresponding thresholds are given by eqs. (3.22), (4.3) and (4.4). On the other hand, beta-function coefficients read: $b_{E_8}^{N=1} = -60$, $b_{E_6}^{N=1} = 36$, $b_{SU(2)}^{N=1} = 12$, $b_{U(1)}^{N=1} = 72$ and $b_{E_8}^{N=2} = -30$, $b_{E_6}^{N=2} = b_{SU(2)}^{N=2} = b_{U(1)}^{N=2} = 42$, and

$$
d_{E_8} = -60 + 960\, q^{\frac{1}{2}}\, \bar{q}^{\frac{1}{2}} - 240\, q + 14040\, \bar{q} + 245760\, q^{\frac{1}{4}}\, \bar{q}^{\frac{5}{4}} + \mathcal{O}\left(q^2\right) ,
$$

$$
\begin{aligned}
d_{E_6} &= 36 + 768\, q^{\frac14}\, \bar q^{\frac14} + 144\, q + 14424\, \bar q + 2496\, q^{\frac12}\, \bar q^{\frac12} + 1536\, q^{\frac54}\, \bar q^{\frac14} \\
&\quad + 247296\, q^{\frac14}\, \bar q^{\frac54} + \mathcal{O}\left(q^2\right) , \\
d_{SU(2)} &= 12 - 8\, \frac{q^{\frac12}}{\bar q^{\frac12}} + 1024\, q^{\frac14}\, \bar q^{\frac14} + 48\, q + 8056\, \bar q + 2048\, q^{\frac54}\, \bar q^{\frac14} + 313344\, q^{\frac14}\, \bar q^{\frac54} + \mathcal{O}\left(q^2\right) , \\
d_{U(1)} &= 72 + 12\, \frac{q^{\frac12}}{\bar q^{\frac12}} + 384\, q^{\frac14}\, \bar q^{\frac14} + 6240\, q^{\frac12}\, \bar q^{\frac12} + 288\, q + 23976\, \bar q + 768\, q^{\frac54}\, \bar q^{\frac14} \\
&\quad + 148224\, q^{\frac14}\, \bar q^{\frac54} + \mathcal{O}\left(q^2\right) , \\
y &= 12 - \frac{1}{q} - 4\, \frac{q}{\bar q} + 16\, \frac{q^{\frac12}}{\bar q^{\frac12}} + 4096\, q^{\frac14}\, \bar q^{\frac14} + 48\, q + 161128\, q^{\frac12}\, \bar q^{\frac12} + 81856\, \bar q \\
&\quad - 4\, \frac{q^2}{\bar q} + 1220608\, q^{\frac14}\, \bar q^{\frac54} + \mathcal{O}\left(q^2\right) .
\end{aligned} \tag{4.14}
$$

Again, numerical evaluation performed with the same accuracy as before leads to:

$$
\delta_{E_8} \approx -6.6 \,,\quad \delta_{E_6} \approx 4.0 \,,\quad \delta_{SU(2)} \approx 6.9 \,,\quad \delta_{U(1)} \approx 0.4 \,. \tag{4.15}
$$

In this case the decomposition (4.10), which was usually adopted in the literature [18], does not hold for the $N = 1$ contributions.

Putting together eqs. (4.1), (4.3), (4.4) and (4.6), we obtain for the threshold corrections of the Z_4 orbifold:

$$
\Delta_i = b_i^{N=1} \log \frac{2\, e^{1-\gamma}}{\pi \sqrt{27}} + \delta_i + b_i^{N=2}\, \Delta\left(T_3, U_3, \overline{T}_3, \overline{U}_3\right) - \frac12\, Y\left(T_3, U_3, \overline{T}_3, \overline{U}_3\right) , \tag{4.16}
$$

where Δ and Y are given by eqs. (3.17) and (3.23) respectively. The decomposition (3.16) where b_i are the full beta-function coefficients is no longer valid, and thus it is not possible to define a unification scale common to all couplings. In order to gain insight it is however interesting to determine the scale $M_U^{E_8-E_6}$ where the E_8 and E_6 gauge couplings meet. This scale can be found by following steps similar to those introduced above. It is expressed as a function of the moduli as well as of the common value of the couplings at that scale: $g_{E_8}\left(M_U^{E_8-E_6}\right) = g_{E_6}\left(M_U^{E_8-E_6}\right) = g_U^{E_8-E_6}$. The latter is related to g_{string} as usually, in a moduli-dependent way. Again, the minimum of this unification scale is reached at $T_3 = U_3 = i$ with the result:

$$
M_{U\ \min}^{E_8-E_6} \approx 5.49 \times 10^{17} \times g_U^{E_8-E_6} \times \frac{1}{\sqrt{1 + 0.08 \times \left(g_U^{E_8-E_6}\right)^2}}\ \text{GeV} \,. \tag{4.17}
$$

Furthermore, eq. (1.1) enables us to compute the splittings of the $SU(2)$ and $U(1)$ gauge couplings with respect to the E_8 and E_6 ones, at that scale. We obtain:

$$
\frac{1}{g^2_{SU(2)}\left(M_U^{E_8-E_6}\right)} - \frac{1}{g^2_{E_8}\left(M_U^{E_8-E_6}\right)} \approx 0.035 \tag{4.18}
$$

and

$$
\frac{1}{g^2_{E_8}\left(M_U^{E_8-E_6}\right)} - \frac{1}{g^2_{U(1)}\left(M_U^{E_8-E_6}\right)} \approx 0.048 \,, \tag{4.19}
$$

which show that the relative splittings are of the order of 4 to 7%.

5. Conclusions

Let us now summarize our results. By using a method introduced in [12] that allows us to handle the infra-red problems, we have determined the complete one-loop gauge coupling corrections (eq. (2.26)) for general heterotic four-dimensional models with at least $N = 1$ supersymmetry. These corrections contain both universal and group-factor dependent terms. Our results for the latter are in agreement with those obtained previously following a different procedure [3, 4], when evaluated within the same ultraviolet renormalization scheme, here the $\overline{DR}$ scheme. This shows that the relation between the running gauge couplings of the low-energy field theory and the string coupling does not depend on the infra-red regularization prescription. It amounts to the decoupling of the (infinite tower of) massive states and allows for an unambiguous definition of string effective theory. Such a conclusion could not have been drown without using a consistent infra-red regulator. Although our result has been established in the framework of an infra-red regulator induced by a particular four-dimensional curved background, we would have reached the same conclusions within any other background possessing similar properties, such as those listed in [32].

Going beyond what has been achieved in previous studies [13], we have determined the moduli-dependent universal part $Y\left(T,U,\overline{T},\overline{U}\right)$ of the thresholds for the class of $N = 2$ four-dimensional theories that come from torus compactification of six-dimensional $N = 1$ ground states. We have obtained an explicit formula for these thresholds (eq. (3.18)) which, thanks to the relation between gauge and R^2-term renormalizations, turns out to be related to the quantity $N_H - N_V$. This is *fully determined* as a consequence of *the anomaly cancellation* (gauge, gravitational and mixed) in the underlying six-dimensional theory. Therefore, the whole class of models under consideration have equal universal thresholds. This implies in particular that the results of [17] are actually more general, and provide the prepotential for all $N = 2$ ground states that are toroidal compactifications of six-dimensional $N = 1$ vacua.

Using the method of orbits of the modular group, we have recasted the integral representation of the above thresholds (3.22) as a power series expansion (eq. (B.11)). This allowed us to analyse the singularity behaviour of $Y\left(T,U,\overline{T},\overline{U}\right)$: although this function is continuous inside the moduli space (in contrast to what was believed), its Laplacian, which is the one-loop correction to the Kähler metric for moduli fields, diverges logarithmically around enhanced-symmetry lines, but remains free of δ-function singularities. Finally, we have used the series representation for analysing the asymptotics of the $N = 2$ universal thresholds (eqs. (B.12) and (B.14)). The leading behaviour is linear with respect to each radius. This blow up as well as the bad behaviour of the group-dependent contributions lead to the well known decompactification problem in string theory. This problem is cured by considering a class of $N = 4$ four-dimensional models, in which two supersymmetries are spontaneously broken [24]. Such models can be thought of as freely acting orbifolds where a translation is performed in the $N = 2$ invariant plane. Their behaviour at large radii is drastically different from the standard orbifolds. Indeed, the $N = 4$ supersymmetry is restored in the decompactification limit. Due to this restoration of the full supersymmetry, the linear divergence of couplings with respect to the radii is absent and this provides a solution to the decompactification problem. It is finally interesting to observe that models

where the $N = 4$ supersymmetry is spontaneously broken down to $N = 2$ and $N = 1$ do also exist [25]. For these, string-string dualities [33] and D-brane technology [36] allow for a better understanding of the mass spectrum and the multiplicities of the perturbative as well as non-perturbative BPS states [37]. This knowledge might eventually help for deriving the structure of the non-perturbative threshold corrections in physically interesting $N = 2$ and $N = 1$ ground states [25].

The results we obtained for the full threshold corrections enabled us to analyse systematically the unification properties of various models. Concerning the $N = 2$ ground states that we have studied in section 3, we observed that the presence of the universal thresholds Y (eq. (3.23)) leads to a lowering of the natural unification scale with minima reached at the self-dual points $T = U = i, \rho$, and given in (3.26). More precisely, for $g_U^2 = \frac{1}{2}$ we have a 5% decrease while for $g_U^2 = 1$ we can reach 10%, with respect to the case where these corrections are not taken into account.

The case of $N = 1$ models is phenomenologically more interesting. In our study we analysed $N = 1$ orbifold constructions. In this case, the above achievements can be used in order to determine analytically the ($N = 2$)-sector contributions to the thresholds with the results (4.3) and (4.4). Concerning the contributions originated from the $N = 1$ sectors, no general formula is available for the moment, except for the Green–Schwarz term, which appears naturally in the S-frame but does not play any role in string unification, as we showed in section 2 by using the results of appendix A. We therefore restricted our attention to the particular cases of Z_3 and Z_4 symmetric orbifolds, which allowed us to draw some interesting conclusions. For the Z_3 orbifold, the decomposition (4.10) makes it possible to define a unification scale for all couplings (eq. (4.12)), similar to the one that we introduced here above in the case of $N = 2$ models (eq. (3.24)), but moduli-independent. Again the presence of universal thresholds reduces this scale by a few percent. However, this decomposition is accidental and does not apply in more general situations, as e.g. the Z_4 orbifold, in contrast to some general wisdom. Thus one cannot any longer define a common unification scale for all couplings. It is however possible to introduce a scale where a pair of couplings meet. In the case of the Z_4 orbifold, we determined that scale for the E_8 and E_6 couplings, and observed that the relative splittings of the others were of the order of 4 to 7%. This situation has to be compared to what happens in ordinary grand unified theories where, it is always possible to chose a scheme, namely the $\overline{DR}$ scheme, such that all couplings are unified at some scale [38].

Despite the various effects and contributions that one can advance in order to reduce the string unification scale, we should be aware that in the models considered, this scale generally concerns groups that have little to do with phenomenology, and that one has somehow to break one of these groups, say E_6, down to some subgroup, eventually leading to the standard model. In order to describe such a realistic situation in the framework of strings, it seems difficult to avoid the introduction of Wilson lines [19]. Those will enhance the moduli space and allow for a better exploration of the various symmetry-breaking possibilities.

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Appendix A: The Green–Schwarz term in $N = 1$ string ground states

We present here a discussion on the appearance of the Green–Schwarz term in $N = 1$ string ground states [5] when string theory results in the dilaton frame are matched with effective supergravity calculations in the S-frame.

We will start with the tree-level plus one-loop bosonic action of the heterotic string in a generic $N = 1$ ground state, $S = S_{\rm tree} + S_{\rm one\ loop}$, where (we set $\alpha' = 1$)

$$S_{\rm tree} = \int d^4x\,\sqrt{G}\,e^{-2\Phi}\left(\frac{1}{2}\left(R + 4(\partial\Phi)^2 - \frac{1}{12}H^2\right) - \frac{k}{4}F^2 - K^{(0)}_{T_\alpha\overline{T}_\beta}\partial T_\alpha\partial\overline{T}_\beta + \cdots\right)\,, \quad \text{(A.1)}$$

$$S_{\rm one\ loop} = \int d^4x\,\sqrt{G}\left(-K^{(1)}_{T_\alpha\overline{T}_\beta}\partial T_\alpha\partial\overline{T}_\beta - \frac{kZ_F}{4}F^2 + \frac{Z_{F\widetilde{F}}}{4}F\widetilde{F} + \frac{1}{2}H^\mu X_\mu + \cdots\right)\,. \quad \text{(A.2)}$$

Here we included a single gauge field, the moduli and the universal sector. As usual

$$H^\sigma = \frac{1}{3!}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{G}}H_{\mu\nu\rho}\,,\quad \tilde{F}^{\mu\nu} = \frac{1}{2!}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{G}}F_{\rho\sigma}\,, \quad \text{(A.3)}$$

and X_μ is a vector that depends on the moduli. The coupling of the antisymmetric tensor to the moduli that arises at one loop is a direct descendant of the anomaly-cancelling Green–Schwarz term in ten dimensions. Going to the Einstein frame where $g_{\mu\nu} = e^{-2\Phi}G_{\mu\nu}$, and introducing the axion by[13]

$$e^{-4\Phi}H^{\mu\nu\rho} = \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}\left(\partial_\sigma A + X_\sigma\right)\,, \quad \text{(A.4)}$$

we obtain the dual action:

$$\widetilde{S}_{\rm tree\ \&\ one\ loop} = \int d^4x\,\sqrt{g}\left(\frac{1}{2}R - (\partial\Phi)^2 - \left(K^{(0)}_{T_\alpha\overline{T}_\beta} + e^{2\Phi}K^{(1)}_{T_\alpha\overline{T}_\beta}\right)\partial T_\alpha\partial\overline{T}_\beta \right.$$
$$\left. - \frac{1}{4}e^{4\Phi}(\partial A + X)^2 - \frac{k}{4}\left(e^{-2\Phi} + Z_F\right)F^2 + \frac{1}{4}\left(kA + Z_{F\widetilde{F}}\right)F\widetilde{F}\right)\,. \quad \text{(A.5)}$$

At tree level the S field is simply $S = A + ie^{-2\Phi}$ and the tree-level Kähler potential is $K_{\rm tree} = -\log\,{\rm Im}\,S + K^{(0)}(T_\alpha,\overline{T}_\beta)$. At one loop the S field mixes with the moduli and is determined by a Kähler potential of the form [5]

$$K_{\rm tree\ \&\ one\ loop} = -\log\left({\rm Im}\,S + V(T_\alpha,\overline{T}_\beta)\right) + K^{(0)}(T_\alpha,\overline{T}_\beta)\,, \quad \text{(A.6)}$$

[13] Note that the X_σ term is the one-loop correction to the tree-level definition of the axion.

where $K^{(0)}(T_\alpha, \overline{T}_\beta) = -\log\, \mathrm{Im}\, T_\alpha - \cdots$. Compatibility of (A.6) with (A.5) implies that

$$S = A + i\left(e^{-2\Phi} - V\right) \ , \quad X_\mu = i\left(V_{T_\alpha}\partial_\mu T_\alpha - V_{\overline{T}_\beta}\partial_\mu \overline{T}_\beta\right) \ , \quad K^{(1)}_{T_\alpha \overline{T}_\beta} = -V_{T_\alpha \overline{T}_\beta} \, . \tag{A.7}$$

Let us consider now the physical gauge coupling constant. From string theory we can calculate the one-loop corrected ith group factor coupling to be, in the dilaton frame, (see eqs. (2.22) and (2.26))

$$\begin{aligned} \frac{1}{g^2_{i,\,\mathrm{eff}}} &= k_i\, e^{-2\langle\Phi\rangle} + \frac{b_i}{16\,\pi^2}\log\frac{M_s^2}{\mu^2} + \frac{\Delta_i}{16\,\pi^2} + \frac{b_i}{16\,\pi^2}(2+2\gamma) \\ &= k_i\,(\,\mathrm{Im}\, S + V) + \frac{b_i}{16\,\pi^2}\log\frac{M_s^2}{\mu^2} + \frac{\Delta_i}{16\,\pi^2} + \frac{b_i}{16\,\pi^2}(2+2\gamma)\,, \end{aligned} \tag{A.8}$$

where μ is the infra-red scale, Δ_i are the thresholds computed in section 2 (eq. (2.26)) and b_i the full beta-function coefficients of the $N=1$ model: $b_i = \sum_r n_i^r\, T_i(r) - 3T(G_i)$.

In the effective supergravity theory, at tree level, the gauge couplings are given by the imaginary part of a holomorphic function f_i, which in this case is $f_i = k_i\, S$. At one loop we have [5, 6]

$$\begin{aligned} \frac{1}{g^2_{i,\,\mathrm{eff}}} &= k_i\ \mathrm{Im}\, S + \frac{b_i}{16\,\pi^2}\left(\log\frac{\Lambda^2}{\mu^2} - \log\ \mathrm{Im}\, S\right) + \ \mathrm{constant} \\ &\quad + \frac{1}{16\,\pi^2}\left(\mathrm{Im}\, f_i^{(1)} + c_i\, K^{(0)} - 2\sum_r T_i(r)\log\det Z^{(0)}_{r_i}\right) \, . \end{aligned} \tag{A.9}$$

Here $c_i = \sum_r n_i^r\, T_i(r) - T(G_i)$ is the tree-level (moduli-dependent) matrix that multiplies the kinetic terms of matter in the representation r of G_i. There is a scheme-dependent constant in (A.9), which for the $\overline{DR}$ scheme was computed in [3, 13], and turns out to be $\frac{b_i}{16\,\pi^2}(2+2\gamma)$. The calculation in the effective field theory was done with an ultraviolet cut-off at Λ. In [6] the cut-off was set to be the Planck scale $\Lambda = M_P$, which is the natural scale from the point of view of supergravity. It was shown in [14] that in heterotic supersymmetric string vacua the relation of the string scale to the Planck mass does not receive corrections in perturbation theory. Thus

$$M_P = M_s\, e^{-\langle\Phi\rangle} \, . \tag{A.10}$$

However, if this relation is expressed in terms of $\mathrm{Im}\, S$ then it does receive corrections already at one loop, since the relation of $\mathrm{Im}\, S$ and $e^{-2\Phi}$ is modified order by order in perturbation theory. Therefore, to first non-trivial order

$$M_s^2\Big|_{\text{one loop}} = \frac{M_P^2}{\mathrm{Im}\, S + V} \, . \tag{A.11}$$

Choosing $\Lambda = M_P$ does not affect the couplings at one loop. Nevertheless, in order to profit from the non-renormalization theorem we can choose the ultraviolet cut-off for the effective field theory to be defined by (A.11), which effectively resums all relevant higher-loop effects due to the renormalization of the relation between $e^{-2\Phi}$ and $\mathrm{Im}S$. This is a slightly different

scheme than the one adopted in [6]. It is similar to loop computations in QCD, where the judicious choice of the relevant scale for a process resums some higher-order effects.

Comparing now (A.8) and (A.9) we obtain to next-to-leading order in $\mathrm{Im}\, S$:

$$\mathrm{Im}\, f_i^{(1)} = -c_i\, K^{(0)} + 2 \sum_r T_i(r) \log\det Z^{(0)}_{r_i} + 16\, \pi^2 k_i\, V + \Delta_i\,. \tag{A.12}$$

All terms in (A.12) are independent of the S field. Moreover [6] since $f^{(1)}$ is holomorphic due to $N = 1$ supersymmetry, the right-hand side has to be a holomorphic function. Thus the non-holomorphicity of the tree-level kinetic terms has to cancel the one-loop non-holomorphicity of the couplings. This is a reflection of the cancellation of the Kähler anomaly in the effective field theory [5].

One more comment is in order here. One-loop calculations in the effective theory reproduce derivatives of the function V. On the other hand, the S field is defined in terms of V itself. A harmonic shift of $V \to V + h + \bar{h}$, where h is holomorphic, can be absorbed in a holomorphic redefinition of S. Thus the coupling in the S frame is defined up to holomorphic redefinitions. This is important in the context of the threshold corrections generated from $N = 1$ sectors.

Finally, formula (A.9) refers to generic heterotic superstring vacua. If one restricts to orbifold compactifications, and use the extra information they provide for the Kähler potential [5, 6], V can be determined (up to holomorphic redefinitions):

$$V = \frac{1}{16\, \pi^2} \left(\Delta^{GS} + Y \right) \tag{A.13}$$

with

$$\Delta^{GS} = \sum_\alpha \delta_\alpha^{GS} \left(\log\, \mathrm{Im}\, T_\alpha + \log\, \mathrm{Im}\, U_\alpha \right) \tag{A.14}$$

the advertised Green–Schwarz term, where the sum extends over the moduli that are not fixed by the orbifold action, and Y is the universal threshold correction appearing in (1.3) and (2.26). Both Y and Δ^{GS} are universal in that they do not depend on the gauge group factor; however their origin is respectively *stringy* and *field-theoretical*, and they should not be confused. The numerical constants δ_α^{GS} can be calculated from the spectrum of the model. An explicit calculation of these quantities for various orbifold models was presented in [5, 6]. For $N = 2$ models we have $\delta^\alpha_{GS} = 0$ and thus

$$V^{N=2} = \frac{1}{16\, \pi^2} Y \left(T, U, \overline{T}, \overline{U} \right) \tag{A.15}$$

with $Y \left(T, U, \overline{T}, \overline{U} \right)$ given in (3.22). In situations with $N = 1$ supersymmetry, δ^γ_{GS} can be evaluated for specific models [6]. They turn out to vanish for $Z_2 \times Z_2$ orbifolds, while for the symmetric Z_3 orbifold we have $\delta^1_{GS} = \delta^2_{GS} = \delta^3_{GS} = 30$:

$$\Delta^{GS} = 30 \sum_{\alpha=1}^{3} \log\, \mathrm{Im}\, T_\alpha\,. \tag{A.16}$$

In the case of the symmetric Z_4 orbifold, the result is $\delta^3_{GS} = 0$ for the $N = 2$ plane that is left unrotated by the orbifold twist, and $\delta^1_{GS} = \delta^2_{GS} = 30$ for the other two planes. Thus

$$\Delta^{GS} = 30 \sum_{\alpha=1}^{2} \log \, \mathrm{Im} \, T_\alpha \, . \tag{A.17}$$

Our last comment concerns the issue of unification. In section 2, working in the dilaton frame, we introduced a renormalized coupling which plays the role of unification coupling, when unification exists. This coupling is related to $g_{\rm string} = \exp\langle\Phi\rangle$ as shown in eq. (2.28). Moduli dependence enters through Y and propagates to the unification scale when M_s is expressed in terms of M_P (see eq. (2.29)). In the S-frame, the same renormalized coupling can be introduced, however expressed in terms of $\mathrm{Im}\, S$:

$$\begin{aligned} \left.\frac{1}{g^2_{\rm renorm}}\right|_{\rm one\ loop} &= \mathrm{Im}\, S + V - \frac{Y}{16\,\pi^2} \\ &= \mathrm{Im}\, S + \frac{\Delta^{GS}}{16\,\pi^2} \, . \end{aligned} \tag{A.18}$$

All moduli dependence is now contained in Δ^{GS}. However, if one uses eq. (A.18) to recast (A.11) in terms of $g_{\rm renorm}$, we obtain (2.29) as in the dilaton frame, and the moduli dependence appears again through Y. Therefore, eq. (2.29) do not depend on the frame, and so are the conclusions about the unification scale, which turns out to be affected by Y but *is not sensitive to* Δ^{GS}.

Appendix B: Explicit formulas for the universal thresholds and asymptotics

The purpose of this appendix is to evaluate, in terms of a multiple series expansion, the universal thresholds for $N = 2$ models that are toroidal compactifications of $N = 1$ six-dimensional heterotic vacua, and analyse their asymptotic behaviours. Our starting point is eq. (3.23). By using (3.15), we can recast expression (3.23) as follows:

$$\begin{aligned} Y\left(T, U, \overline{T}, \overline{U}\right) &= \frac{1}{12} \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau} \left(\Gamma_{2,2}\left(T, U, \overline{T}, \overline{U}\right) \left(\overline{E}_2 - \frac{3}{\pi\, \mathrm{Im}\,\tau}\right) \frac{\overline{E}_4\, \overline{E}_6}{\bar\eta^{24}} + 264 \right) \\ & -22 \left(\log\left(\left|\eta(T)\right|^4 \left|\eta(U)\right|^4 \mathrm{Im}\, T\, \mathrm{Im}\, U \right) + \log \frac{8\pi\, e^{1-\gamma}}{\sqrt{27}} \right) \\ & + \frac{1}{3} \log \left| j(T) - j(U) \right| . \end{aligned} \tag{B.1}$$

The remaining integral in (B.1) can be evaluated using the results of [17] with [14]:

$$\begin{aligned} I\left(T, U, \overline{T}, \overline{U}\right) &= \frac{1}{12} \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau} \left(\Gamma_{2,2}\left(T, U, \overline{T}, \overline{U}\right) \left(\overline{E}_2 - \frac{3}{\pi\, \mathrm{Im}\,\tau}\right) \frac{\overline{E}_4\, \overline{E}_6}{\bar\eta^{24}} + 264 \right) \\ &= \frac{1}{3} \mathrm{Re} \left(-\, 24 \sum_{k>0} \left(11\, \mathcal{L}i_1\left(e^{2\pi i k T}\right) - \frac{30}{\pi\, \mathrm{Im}\, T\, \mathrm{Im}\, U} \mathcal{P}(kT) \right) \right. \end{aligned}$$

[14]Here we use the convention $\Theta(0) = \frac{1}{2}$.

$$
\begin{aligned}
& -24\sum_{\ell>0}\left(11\,\mathcal{L}i_1\left(e^{2\pi i\ell U}\right)-\frac{30}{\pi\,\mathrm{Im}\,T\,\mathrm{Im}\,U}\,\mathcal{P}(\ell U)\right)\\
& +\sum_{k>0,\,\ell>0}\left(\tilde{c}(k\ell)\,\mathcal{L}i_1\left(e^{2\pi i(kT+\ell U)}\right)-\frac{3\,c(k\ell)}{\pi\,\mathrm{Im}\,T\,\mathrm{Im}\,U}\,\mathcal{P}(kT+\ell U)\right)\\
& +\mathcal{L}i_1\left(e^{2\pi i\left(\,\mathrm{Re}\,T-\,\mathrm{Re}\,U+i|\,\mathrm{Im}\,T-\,\mathrm{Im}\,U|\right)}\right)\\
& -\frac{3}{\pi\,\mathrm{Im}\,T\,\mathrm{Im}\,U}\,\mathcal{P}\Big(\,\mathrm{Re}\,T-\,\mathrm{Re}\,U+i|\,\mathrm{Im}\,T-\,\mathrm{Im}\,U|\Big)\Big)\\
+&\frac{60\,\zeta(3)}{\pi^2\,\mathrm{Im}\,T\,\mathrm{Im}\,U}+22\left(\log(\,\mathrm{Im}\,T\,\mathrm{Im}\,U)+\log\frac{8\pi\,e^{1-\gamma}}{\sqrt{27}}\right)\\
+&\left(\frac{4\pi}{3}\frac{(\,\mathrm{Im}\,U)^2}{\mathrm{Im}\,T}-\frac{22\,\pi}{3}\,\mathrm{Im}\,U-4\pi\,\mathrm{Im}\,T\right)\Theta(\,\mathrm{Im}\,T-\,\mathrm{Im}\,U)\\
+&\left(\frac{4\pi}{3}\frac{(\,\mathrm{Im}\,T)^2}{\mathrm{Im}\,U}-\frac{22\,\pi}{3}\,\mathrm{Im}\,T-4\pi\,\mathrm{Im}\,U\right)\Theta(\,\mathrm{Im}\,U-\,\mathrm{Im}\,T)\,. \qquad \text{(B.2)}
\end{aligned}
$$

Here $c(n)$ and $\tilde{c}(n)$ are the coefficients of the Laurent expansions:

$$\frac{E_4\,E_6}{\eta^{24}}=\sum_{n=-1}^{\infty}c(n)\,q^n=\frac{1}{q}-240-141444\,q-8529280\,q^2+\cdots \qquad \text{(B.3)}$$

and

$$\frac{E_2\,E_4\,E_6}{\eta^{24}}=\sum_{n=-1}^{\infty}\tilde{c}(n)\,q^n=\frac{1}{q}-264-135756\,q-5117440\,q^2+\cdots\,. \qquad \text{(B.4)}$$

The function $\mathcal{P}(x)$ is defined by

$$\mathcal{P}(x)=\,\mathrm{Im}\,x\,\mathcal{L}i_2\left(e^{2\pi ix}\right)+\frac{1}{2\pi}\,\mathcal{L}i_3\left(e^{2\pi ix}\right)\,, \qquad \text{(B.5)}$$

and $\mathcal{L}i_j$ are the polylogarithms

$$
\begin{aligned}
\mathcal{L}i_1(x) &= \sum_{j=1}^{\infty}\frac{x^j}{j}=-\log(1-x)\,, && \text{(B.6)}\\
\mathcal{L}i_2(x) &= \sum_{j=1}^{\infty}\frac{x^j}{j^2}\,, && \text{(B.7)}\\
\mathcal{L}i_3(x) &= \sum_{j=1}^{\infty}\frac{x^j}{j^3}\,. && \text{(B.8)}
\end{aligned}
$$

The above integral I is logarithmically divergent when $T\to U$; the singularity arises from the term $\frac{1}{3}\,\mathrm{Re}\,\mathcal{L}i_1\left(e^{2\pi i\left(\,\mathrm{Re}\,T-\,\mathrm{Re}\,U+i|\,\mathrm{Im}\,T-\,\mathrm{Im}\,U|\right)}\right)$. This very divergence cancels the one appearing in Y through $\frac{1}{3}\log|j(T)-j(U)|$. Indeed we can use the product representation

$$j(T)-j(U)=e^{-2\pi iT}\prod_{k>0,\,\ell>-2}\left(1-e^{2\pi i(kT+\ell U)}\right)^{\hat{c}(k\ell)}\,, \qquad \text{(B.9)}$$

where $\hat{c}(n)$ are defined by[15]

$$j(q) - 744 = \sum_{n=-1}^{\infty} \hat{c}(n)\, q^n = \frac{1}{q} + 196884\, q + 21493760\, q^2 + \cdots \,. \tag{B.10}$$

Putting everything together we obtain:

$$\begin{aligned} Y \;\; = \;\; & \mathrm{Re} \sum_{k>0,\,\ell>0} \left(\frac{\tilde{c}(k\ell) - \hat{c}(k\ell)}{3}\, \mathcal{L}i_1 \left(e^{2\pi i (kT + \ell U)} \right) - \frac{c(k\ell)}{\pi \,\mathrm{Im}\, T \,\mathrm{Im}\, U}\, \mathcal{P}(kT + \ell U) \right) \\ & + \frac{60}{\pi^2 \,\mathrm{Im}\, T \,\mathrm{Im}\, U} \left(\zeta(3) + 4\pi \,\mathrm{Re} \sum_{k>0} \mathcal{P}(kT) + 4\pi \,\mathrm{Re} \sum_{\ell>0} \mathcal{P}(\ell U) \right) \\ & - \frac{1}{\pi \,\mathrm{Im}\, T \,\mathrm{Im}\, U}\, \mathrm{Re}\, \mathcal{P}\Big(\mathrm{Re}\, T - \mathrm{Re}\, U + i |\, \mathrm{Im}\, T - \mathrm{Im}\, U| \Big) \\ & + \left(\frac{4\pi}{3} \frac{(\mathrm{Im}\, U)^2}{\mathrm{Im}\, T} + 4\pi \,\mathrm{Im}\, T \right) \Theta(\mathrm{Im}\, T - \mathrm{Im}\, U) \\ & + \left(\frac{4\pi}{3} \frac{(\mathrm{Im}\, T)^2}{\mathrm{Im}\, U} + 4\pi \,\mathrm{Im}\, U \right) \Theta(\mathrm{Im}\, U - \mathrm{Im}\, T)\,. \end{aligned} \tag{B.11}$$

The above expression enables us to check that Y is finite and continuous at $T = U$ for finite T and U. Concerning $\partial_T \partial_{\overline{T}} Y$, it is obvious that potential singularities may arise from the term $\frac{-1}{\pi \,\mathrm{Im}\, T \,\mathrm{Im}\, U}\, \mathrm{Re}\, \mathcal{P}\Big(\mathrm{Re}\, T - \mathrm{Re}\, U + i |\, \mathrm{Im}\, T - \mathrm{Im}\, U| \Big)$ as well as from the Θ-functions. The latter turn out to give regular terms while the former leads to (3.33).

Finally, by using (B.11), various asymptotic behaviours can be studied. We restrict again to the case where $\mathrm{Re}\, T = \mathrm{Re}\, U = 0$. The limit $R_1 = R_2 = R \to \infty$ was derived in [13] with the result:

$$Y(R) = 4\pi R^2 + \frac{60\,\kappa}{\pi R^2} + \mathcal{O}\left(e^{-\pi R^2} \right) , \tag{B.12}$$

where

$$\kappa = \frac{2}{\pi^2} \zeta(4) + \sum_{j>0} \left(\frac{\coth \pi j}{\pi} \frac{1}{j^3} + \frac{1}{\sinh^2 \pi j} \frac{1}{j^2} \right) \approx 0.61\,. \tag{B.13}$$

Following similar steps one can derive the asymptotic expansions for $\mathrm{Im}\, T,\ \mathrm{Im}\, U \to \infty$ with the ratio kept fixed. This amounts to taking $R_2 \to \infty$, while $R_1 = \sqrt{\mathrm{Im}\, T / \mathrm{Im}\, U}$ is finite. We obtain:

$$Y(R_1, R_2) = \begin{cases} \frac{4\pi}{3} R_2 \left(\frac{1}{R_1^3} + 3\, R_1 \right) \Theta(R_1 - 1) + \frac{4\pi}{3} R_2 \left(R_1^3 + \frac{3}{R_1} \right) \Theta(1 - R_1) \\ + \frac{60}{\pi^2 R_2^2}\, \zeta(3) + \mathcal{O}\left(e^{-\pi R_2^2} \right) \quad \text{if } R_1 \neq 1 \\ \\ \frac{16\,\pi}{3} R_2 + \frac{119}{2\pi^2 R_2^2}\, \zeta(3) + \mathcal{O}\left(e^{-\pi R_2^2} \right) \quad \text{if } R_1 = 1\,, \end{cases} \tag{B.14}$$

from which it appears that Y is not continuous at $T = U$ when the radii become large. Similar results can be reached for $R_1 \to \infty$, R_2 finite; then the discontinuity appears at $R_2 = 1$, corresponding to $T = \frac{1}{U}$.

[15] We use the notation $j(\tau)$ or $j(q)$ with $q = e^{2\pi i \tau}$.

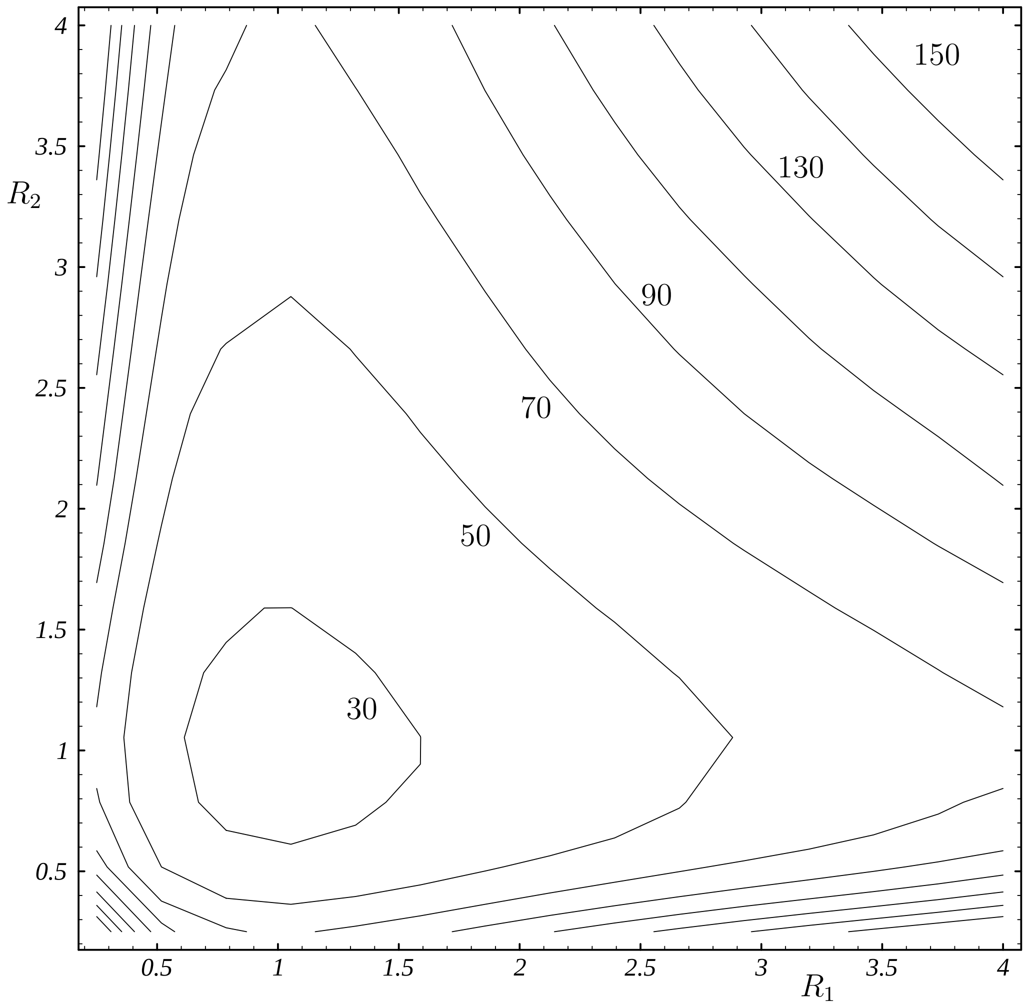

Figure 1: Contour plots of the universal thresholds $Y(R_1, R_2)$ as a function of the internal radii R_1 and R_2.

We have also calculated Y numerically. One can verify that the $\mathrm{Re}\, T$, $\mathrm{Re}\, U$ dependence is very weak and leads to small oscillations of the order $< 2\%$ around the $\mathrm{Re}\, T = \mathrm{Re}\, U = 0$ values. The results for $\mathrm{Re}\, T = \mathrm{Re}\, U = 0$ are presented in figure 1. As expected, there is another minimum at the other self-dual point $T = U = \rho$. It has the same depth as the one at $T = U = i$ discussed above, to the accuracy of its numerical evaluation. We come to the conclusion that $Y(R_1, R_2) \geq Y(1, 1)$ where $Y(1, 1) \approx 24.4$ is the value of the minimum at the point $R_1 = R_2 = 1$.

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